TRINITY COLLEGE

IN NOMINE DOMINI

Semester Two Examination, 2021

Question/Answer booklet

MATHEMATICS METHODS UNITS 3&4

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Your name

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(8 marks)**

An online employment survey on a public internet forum attracted 75 responses from mine workers, of whom 30 said that they were employed on a temporary contract.

(a) Use the survey data to construct a 90% confidence interval for the population proportion of mine workers employed on a temporary contract. (3 marks)

Solution
$0.4 \pm 1.645\sqrt{\dfrac{0.4(1-0.4)}{75}}$ $(0.307, 0.493)$
Specific behaviours
✓ indicates use of correct method   ✓ correct z-value   ✓ calculates interval

(b) Assuming the survey was reliable, determine the sample size required to conduct a follow-up survey so that a 95% confidence interval for the population proportion of mine workers employed on a temporary contract will have a margin of error close to 0.05. (3 marks)

Solution
$n = \dfrac{1.96^2 \times 0.4 \times (1-0.4)}{0.05^2}$ $= 368.78$ Hence sample size of 369 is required.
Specific behaviours
✓ indicates use of correct method   ✓ correct z-value   ✓ calculates integer sample size

(c) Identify and explain a possible source of bias that may arise from this type of survey. (2 marks)

Solution
Identify: volunteer, convenience, etc., sampling.   Explanation: likely to be biased because it is not a true random sample drawn from the population.
Specific behaviours
✓ identifies possible source of bias   ✓ explains why it is a possible source of bias

Question 10 **(8 marks)**

A capacitor in a circuit starts to discharge. The voltage V across the capacitor after t milliseconds is changing at a rate given by

$$\frac{dV}{dt} = \frac{-162}{(2t+3)^2}, \qquad t \geq 0.$$

(a) Calculate the initial rate of change of voltage. (1 mark)

Solution
$V'(0) = -\frac{162}{9} = -18 \text{ V/ms}$
Specific behaviours
✓ correct rate

(b) Determine the change in voltage during the third millisecond. (3 marks)

Solution
$\Delta V = \int_2^3 \frac{-162}{(2t+3)^2} dt$ $= -\frac{18}{7} \approx -2.571 \text{ V}$
Specific behaviours
✓ indicates correct interval of time   ✓ writes correct integral   ✓ correct change

(c) Given that the initial voltage across the capacitor was 26 volts, determine the time for the voltage to fall to 1 volt. (4 marks)

Solution
$\Delta V = 1 - 26 = -25$ $\Delta V = \int_0^T \frac{-162}{(2t+3)^2} dt$ $= \frac{81}{2T+3} - 27$ Hence require $\frac{81}{2T+3} - 27 = -25$ $T = \frac{75}{4} = 18.75 \text{ ms}$
Specific behaviours
✓ indicates required change in voltage   ✓ expression for change in V ✓ forms equation   ✓ calculates time

Question 11 **(7 marks)**

A company packages tea in tins marked with a net weight of 375 g. The weight of tea in the tins is normally distributed with a mean of 384.5 g and a standard deviation of 4.8 g.

(a) Determine the proportion of tins that contain more than the marked weight. (2 marks)

Solution
$X \sim N(384.5, 4.8^2)$ $P(X > 375) = 0.9761$
Specific behaviours
✓ states expression for probability ✓ correct probability

(b) What proportion of tins containing more than the marked weight contain more than 390 g of tea? (2 marks)

Solution
$P(X > 390 \mid X > 375) = \dfrac{P(X > 390)}{P(X > 375)}$ $= \dfrac{0.1259}{0.9761} = 0.1290$
Specific behaviours
✓ states expression for conditional probability ✓ calculates probability

(c) The company has decided that no more than 1 in 500 tins should contain less than the marked weight of tea. To achieve this, they will pack more tea in each tin and hence increase the mean of the distribution whilst maintaining the existing standard deviation. Determine the minimum increase in the mean required. (3 marks)

Solution
$P(Z < z) = \dfrac{1}{500} \Rightarrow z = -2.8782$ $\dfrac{375 - \mu}{4.8} = -2.8782$ $\mu = 388.8$ $388.8 - 384.5 = 4.3$ Extra weight of tea is 4.3 g.
Specific behaviours
✓ indicates z-score ✓ equation for mean ✓ solves for mean and states increase

Question 12 **(8 marks)**

A factory makes identical plastic key fobs in four different colours. 15% are red, 20% are green, 25% are blue and the remainder orange. The key fobs are randomly packed into boxes of 120.

Quality control at the factory randomly sample several boxes from the production line daily and record, amongst other things, the proportion of green key fobs in each box.

(a) Describe the continuous probability distribution that the sample proportion of green key fobs will approximate over time, including any parameters. (3 marks)

Solution
$v = \dfrac{0.2 \times (1 - 0.2)}{120} \approx 0.001\overline{3}$ $s = \sqrt{v} \approx 0.0365$ The sample proportions will approximate a normal distribution with mean of 0.2 and variance of 0.00133 (or standard deviation of 0.0365).
Specific behaviours
✓ indicates normal distribution ✓ correct mean ✓ correct variance (or standard deviation)

(b) Use a similar continuous probability distribution to that in part (a) to calculate an approximation for the probability that the proportion of orange key fobs in a randomly chosen box is at least 35%. (3 marks)

Solution
$1 - 0.15 - 0.2 - 0.25 = 0.4$ $X \sim N(0.4, 0.002) \sim N(0.4, 0.08765^2)$ $P(X \geq 0.35) = 0.868$
Specific behaviours
✓ indicates proportion ✓ defines sampling distribution ✓ calculates probability

(c) Briefly explain why the distribution in part (a) is an approximation and state the key factor that determines the closeness of the approximation. (2 marks)

Solution
The true distribution of proportions is binomial. The larger the sample size (n), the closer the normal distribution approximates the binomial distribution.
Specific behaviours
✓ states true distribution ✓ states sample size as key factor

Question 13 **(7 marks)**

A small body starts from rest at point A and moves in a straight line until it reaches point B, where it is again stationary.

The acceleration of the body t seconds after leaving A is a m/s^2, where $a = 0.018t - 0.0003t^2$.

Determine

(a) the time taken for the body to travel from A to B. (3 marks)

Solution
$$v(t) = \int a\,dt = 0.009t^2 - 0.0001t^3 + c$$ $$v(0) = 0 \Rightarrow c = 0$$ $$v(t) = 0 \Rightarrow t = 0, 90$$ Hence body took 90 seconds to travel from A to B.
Specific behaviours
✓ indicates use of integration to obtain velocity ✓ expression for $v(t)$ ✓ solves $v(t) = 0$ and states travel time

(b) the distance from A to B. (2 marks)

Solution
Since $v(t) \neq 0$ within interval, then: $$d = \int_0^{90} v(t)\,dt = 546.75\text{ m}$$
Specific behaviours
✓ writes integral ✓ correct distance

(c) the maximum velocity of the body between A and B. (2 marks)

Solution
Maximum velocity when $a = 0$: $$a(t) = 0 \Rightarrow t = 0, 60$$ $$v(60) = 10.8\text{ m/s}$$
Specific behaviours
✓ indicates time ✓ correct velocity

Question 14 **(8 marks)**

The level of atmospheric carbon dioxide C in parts per million was measured by scientists at an Arctic base and was observed to increase from 322.9 ppm on 1 January 1967, to 335.4 ppm by 1 January 1976.

The level can be modelled by equation $C = C_0 e^{kt}$, where t is the number of years from the start of the year 1960.

(a) Determine an expression for the constant k in the form $a\ln(b)$ and hence show that its value is approximately 0.00422. (3 marks)

Solution
$1976 - 1967 = 9$ years $e^{9k} = \dfrac{335.4}{322.9}$ $k = \dfrac{1}{9}\ln\left(\dfrac{3354}{3229}\right)$ $= 0.00422$
Specific behaviours
✓ indicates correct time interval ✓ uses ratio of values ✓ correct expression for k and evaluates

(b) Determine the value of the constant C_0. (2 marks)

Solution
$322.9 = C_0 e^{0.00422(7)}$ $C_0 = 313.5$
Specific behaviours
✓ substitutes into equation ✓ solves for C_0

(c) Calculate the level of atmospheric carbon dioxide at the start of the year 1995. (1 mark)

Solution
$C(35) = 363.4$ ppm
Specific behaviours
✓ correct value

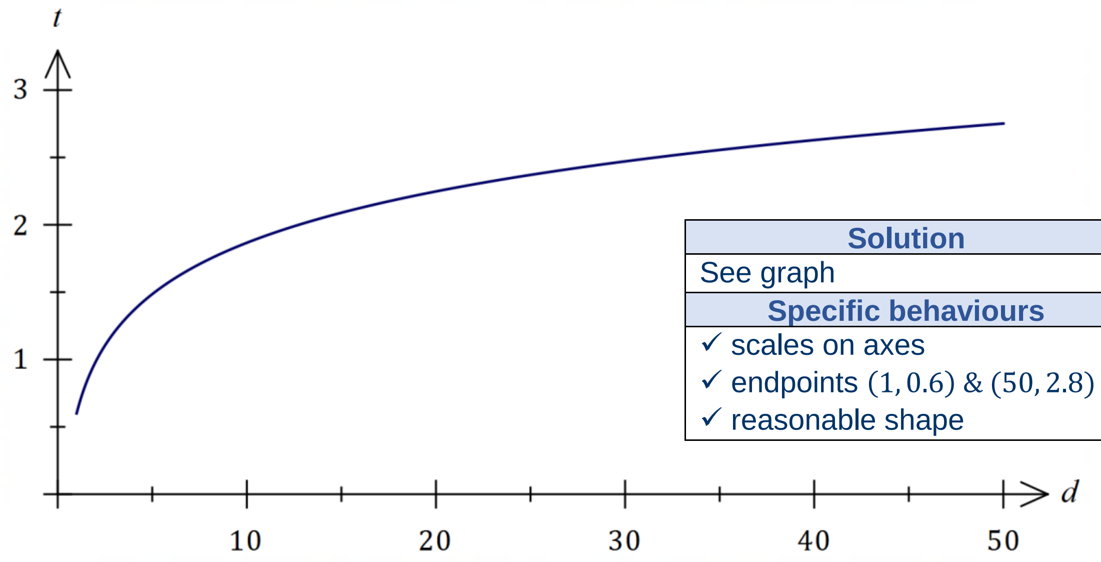
(d) Determine the rate at which the level of atmospheric carbon dioxide was increasing at the start of the year 1995. (2 marks)

Solution
$\Delta C = 363.4 \times 0.00422$ $= 1.53$ ppm/yr
Specific behaviours
✓ indicates method ✓ calculates correct rate

Question 15 **(7 marks)**

The time, t seconds, for a trained rat to pick a bead out of a container and drop it into a small hole when the distance of the bead container from the hole was d cm can be modelled by the relationship $t = 0.6 + 0.55\ln(d)$ for $d \geq 1$.

(a) Sketch the graph of t as a function of d for $1 \leq d \leq 50$ cm. (3 marks)



Solution
See graph
Specific behaviours
✓ scales on axes
✓ endpoints $(1, 0.6)$ & $(50, 2.8)$
✓ reasonable shape

(b) Determine the extra time taken by the rat to move a bead when the distance of the bead container from the hole increases from 20 cm to 60 cm. (1 mark)

Solution
$t(60) - t(20) = 2.852 - 2.248$ $= 0.604\text{ s}$
Specific behaviours
✓ calculates change

(c) Use the relationship to show that if the distance of the bead container from the hole increases from x cm to $3x$ cm, the change in time is constant. (3 marks)

Solution
$t(x) = 0.6 + 0.55\ln(x)$ $t(3x) = 0.6 + 0.55\ln(3x)$ $= 0.6 + 0.55(\ln 3 + \ln x)$ $= 0.6 + \mathbf{0.55\ln 3} + 0.55\ln x$ $\Delta t = t(3x) - t(x)$ $= 0.55\ln 3$ Hence change in time is a constant.
Specific behaviours
✓ expressions for $t(x)$ and $t(3x)$
✓ isolates bolded term from $t(3x)$
✓ calculates change and deduces constant

Alternative Solution
$\Delta t = \int_x^{3x} \frac{dt}{dd}\,dd$ $= \int_x^{3x} \frac{0.55}{d}\,dd$ $= \frac{11\ln 3}{20}$ Hence change in time is a constant.
Specific behaviours
✓ integral for total change from rate of change
✓ expression for rate of change
✓ calculates change and deduces constant

Question 16 **(8 marks)**

A person drives to work n times each month and on any one journey, the probability that they arrive late for work is p.

(a) When $n = 18$ and $p = 0.2$ determine the probability that

(i) they are never late for work in a month. (2 marks)

Solution
$X \sim B(18, 0.2)$ $P(X = 0) = 0.0180$
Specific behaviours
✓ indicates binomial distribution ✓ correct probability

(ii) they are late for work at least twice in a month. (1 mark)

Solution
$P(X \geq 2) = 0.9009$
Specific behaviours
✓ correct probability

(iii) they are late for work at least once in each of six consecutive months. (2 marks)

Solution
$1 - 0.0180 = 0.9820, \quad 0.9820^6 = 0.8967$ or $Y \sim B(6, 0.9820), \quad P(Y = 6) = 0.8967$
Specific behaviours
✓ indicates appropriate method ✓ correct probability

(b) Determine n and p when the mean and variance of the number of times the person is late for work each month is 3.6 and 3.06 respectively. (3 marks)

Solution
$np = 3.6, \quad np(1 - p) = 3.06$ Solve simultaneously: $n = 24, \quad p = 0.15$
Specific behaviours
✓ equation for mean ✓ equation for sd or variance ✓ correct values

Question 17 **(8 marks)**

The local newspaper in a large city claimed that over 60% of the city's population trusted them. To check this claim, a research group took a random sample of 975 people in the city and found that 546 of them trusted the newspaper.

(a) Construct a 95% confidence interval for the proportion of all people in the city who trust the newspaper and hence comment on the validity of the newspaper's claim. (4 marks)

Solution
Calculation:
$$0.56 \pm 1.96 \times \sqrt{\frac{0.56(1-0.56)}{975}}$$
Interval:
$$(0.529, 0.591)$$
The claimed proportion of 0.6 made by the newspaper is not contained in the 95% confidence interval and hence the claim is unlikely to be valid.
Specific behaviours
✓ indicates correct calculation ✓ correct interval ✓ states claimed proportion not in interval ✓ states claim not valid

(b) The research group carried out the same sampling task in different city, from which the 99% confidence interval $(0.594, 0.708)$ was constructed. Determine the number of people in this sample who trusted their local newspaper. (4 marks)

Solution
$$z_{0.99} = 2.576$$ $$E = (0.708 - 0.594) \div 2 = 0.057$$ $$p = 0.594 + 0.057 = 0.651$$ $$n = \frac{2.576^2 \times 0.651 \times (1-0.651)}{0.057^2} = 464$$ $$x = 464 \times 0.651 = 302 \text{ people}$$
Specific behaviours
✓ uses correct z-score ✓ indicates margin of error and proportion ✓ calculates sample size ✓ calculates number of people

Question 18 **(6 marks)**

A player throws a regular tetrahedral die whose faces are numbered $1, 2, 3$ and 4. If the player throws a two, the die is thrown a second time, and in this case the score is the sum of 2 and the second number; otherwise, the score is the number obtained. The player has no more than two throws. Let X be the random variable denoting the player's score.

(a) Write down the probability distribution of X. (3 marks)

Solution

x	1	3	4	5	6
$P(X=x)$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

Specific behaviours

✓ correct x values
✓ probabilities sum to 1
✓ correct probabilities

(b) Determine the mean and standard deviation of X. (2 marks)

Solution

$$E(X) = \frac{25}{8} = 3.125$$

$$\text{Var}(X) = \frac{135}{64} \Rightarrow sd = \frac{3\sqrt{15}}{8} \approx 1.452$$

Specific behaviours

✓ mean
✓ standard deviation

(c) Determine $P\big(X = 1 \mid X \le E(X)\big)$. (1 mark)

Solution

$$P = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{5}{16}} = \frac{4}{9}$$

Specific behaviours

✓ correct probability

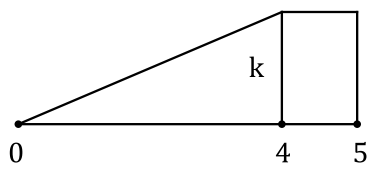
Question 19 **(8 marks)**

An electronic device is powered by an AAA battery that will always last for a minimum of 12 hours. The random variable T is the number of hours exceeding 12 for which the device will continue to operate, and it has probability density function f shown below:

$$f(t) = \begin{cases} \dfrac{kt}{4} & 0 \le t \le 4 \\ k & 4 < t \le 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of the constant k. (3 marks)

Solution



$$\text{Area} = \frac{1}{2} \times 4 \times k + k = 3k$$

Since $f(t)$ is a pdf then $3k = 1 \Rightarrow k = \frac{1}{3}$.

Specific behaviours

✓ sketch of function
✓ calculates area in terms of k
✓ correct value of k

(b) Calculate the mean of T. (2 marks)

Solution

$$E(T) = \int_0^4 \frac{t^2}{12} dt + \int_4^5 \frac{t}{3} dt$$
$$= \frac{16}{9} + \frac{3}{2} = \frac{59}{18} = 3.2\overline{7}\text{ h}$$

Specific behaviours

✓ correct expression for mean
✓ calculates mean

(c) Given that $P(T \ge a) = 0.865$, determine the value of the constant a. (3 marks)

Solution

$$P(T \ge a) = 0.865 \Rightarrow \int_0^a \frac{t}{12} dt = 1 - 0.865$$

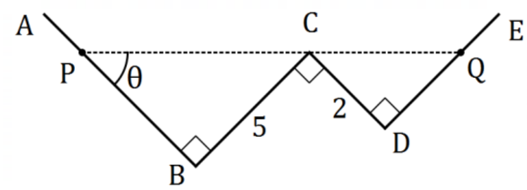
$$\frac{a^2}{24} = 0.135, \quad a^2 = 3.24, \quad a = 1.8$$

Specific behaviours

✓ integral with a as a bound / indicates use of triangle area
✓ evaluates integral a / forms equation
✓ positive value of a

Question 20 **(8 marks)**

In the diagram, $ABCDE$ is part of a plan of the interior of an office, where walls BC and CD have lengths 5 and 2 m respectively. An electrician must run a cable from point P on wall AB in a straight line to point Q on wall DE so that it just touches corner C. The angle the cable PQ makes with wall AB is θ.



(a) When $\theta = 32°$, show that the length L of the cable is approximately 11.8 m. (2 marks)

Solution

$$PC = \frac{5}{\sin 32°}, \qquad CQ = \frac{2}{\cos 32°}$$

$$L = 9.44 + 2.36 = 11.8\text{ m}$$

Specific behaviours

✓ indicates expressions for PC and CQ
✓ evaluates and sums

(b) Use a calculus method to determine the optimum angle that minimises the length L of the cable, state what this minimum length is and justify that it is a minimum. (6 marks)

Solution

$$L = \frac{5}{\sin\theta} + \frac{2}{\cos\theta}, \qquad \frac{dL}{d\theta} = -\frac{5\cos\theta}{\sin^2\theta} + \frac{2\sin\theta}{\cos^2\theta}$$

$$\frac{dL}{d\theta} = 0 \Rightarrow \frac{5\cos\theta}{\sin^2\theta} = \frac{2\sin\theta}{\cos^2\theta}$$

$$\frac{5}{2} = \frac{\sin^3\theta}{\cos^3\theta} = \tan^3\theta\,, \qquad \theta = \tan^{-1}\sqrt[3]{\left(\frac{5}{2}\right)} \approx 0.936^r \approx 53.6°$$

$$L(0.936) = 9.58\text{ m}$$

Justify minimum:
Either sign test $L'(0.93) \approx -0.17,\ L'(0.94) \approx 0.12$
Hence 9.58 is the minimum length as the gradient changes from −ve to +ve as θ increases through 0.936.

Or second derivative $L''(0.936) \approx 28.7$
Hence 9.58 is the minimum length as the function is concave up when $\theta = 0.936$.

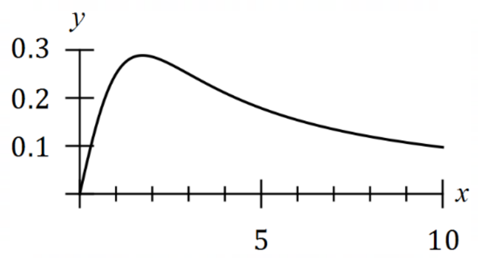
Specific behaviours

✓ expression for length
✓ writes first derivative
✓ equates first derivative to 0 and solves for θ
✓ calculates minimum length
✓ indicates use of second derivative / sign test
✓ justifies length minimum

Question 21 **(7 marks)**

The graph of $y = f(x)$ is shown, where

$$f(x) = \frac{x}{3 + x^2}, \qquad x \geq 0.$$



$f(x)$ is concave down for $0 < x < 3$.

(a) Determine the area bounded by the graph of f and the line $y = \frac{x}{12}$. (3 marks)

Solution
$\frac{x}{3 + x^2} = \frac{x}{12} \Rightarrow x = 0, 3$ $\int_0^3 \frac{x}{3 + x^2} - \frac{x}{12} dx = \ln(2) - \frac{3}{8} \approx 0.318$ sq units
Specific behaviours
✓ solves for bounds   ✓ writes integral   ✓ calculates area

The line $y = mx$ and the graph of f enclose a finite region R.

(b) Determine the values of the slope m for which R exists. (2 marks)

Solution
$f'(0) = \frac{1}{3}$ Hence $0 < m < \frac{1}{3}$
Specific behaviours
✓ slope at origin   ✓ correct inequality

(c) Determine the area of R in terms of m. (2 marks)

Solution
$\frac{x}{3 + x^2} = mx \Rightarrow x = \sqrt{\frac{1 - 3m}{m}}$ $\int_0^{\sqrt{\frac{1-3m}{m}}} \frac{x}{3 + x^2} - mx\, dx = \frac{3m - \ln(m) - \ln(3) - 1}{2}$
Specific behaviours
✓ integral, with bounds   ✓ simplified expression for area

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

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